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Effective Conductivity and Skew Brownian Motion

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We consider heat conduction in a periodic body which is composed of finitely many different components. The effective conductivity is represented in terms of skew Brownian motion. The representation formula is a fluctuation-dissipation relation. The dissipation term in this formula is related to the transmission of heat through the surface separating the different components of the body; it is described by the skew reflections of Brownian motion at these surfaces. The problems caused by the discontinuity of the microscopic conductivity are handled in the framework of Dirichlet forms.

KEY WORDS: Effective conductivity; Green–Kubo formula; discontinuous random media; skew Brownian motion; reversible diffusions; Dirichlet forms.

INTRODUCTION

We consider a heat-conducting body in Euclidean space $\mathbf{R}^d$, $d \geqslant 1$. Let $a(x)$ be the conductivity in $x \in \mathbf{R}^d$, where $a\colon \mathbf{R}^d \to [0, \infty)$ is a given function varying on a microscopic scale. Suppose the body is periodic and composed of finitely many different components, i.e., we assume the function a is periodic and takes on finitely many different values. The conductivity of the body in the average, the so-called *effective conductivity*, is a constant independent of x. It is given by a tensor $\hat{a} = (\hat{a}_{ij})_{1 \leqslant i,j \leqslant d}$, which is roughly defined as follows. Denoting by e_k $(1 \leqslant k \leqslant d)$ the unit vectors of an orthonormal basis in $\mathbf{R}^d$, one has

$$\hat{a}_{ij} = \text{average heat flux in direction } e_j \text{ produced by an average negative temperature gradient } e_i \ (1 \leqslant i, j \leqslant d) \tag{0.1}$$

The problem of getting a managable expression for $\hat{a}$ has a long history; see, for example, ref. 11, where the literature up to the year 1926 is

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surveyed. More recently a spectral representation of the effective conductivity has been found[7]; furthermore, in the context of the general theory of homogenization of differential operators[17] a variational characterization of $\hat{a}$ is known. From this variational principle one easily obtains, for example, that $\hat{a}$ is bounded below by the harmonic mean and above by the arithmetical mean of the microscopic conductivity. However, it is not easy to visualize geometrically what kind of mean $\hat{a}$ precisely is and how it results from the heat transmission through the surfaces which separate the different components of the body. In this paper we want to give such a visualization by means of an explicit representation of $\hat{a}$ in terms of the paths of Brownian motion.

To explain this representation we consider the heat equation

$$\frac{\partial}{\partial t} u(t, x) + \frac{1}{2} \operatorname{div}(-a(x) \cdot \operatorname{grad} u(t, x)) = 0 \qquad (t > 0, x \in \mathbf{R}^d) \tag{0.2}$$

with a random (not necessarily periodic, only ergodic) conductivity $\{a(x), x \in \mathbf{R}^d\}$. Under smoothness assumptions on $a(x)$ it is known[14, 12] that the diffusion process corresponding to (0.2) has an effective diffusivity which coincides with the effective conductivity defined by (0.1). In the case of smooth conductivities it was furthermore observed in ref. 5 that time reversibility of the diffusion process entails a *Green–Kubo* formula (cf. pp. 177 in ref. 16) for the effective diffusivity.

In the following we want to prove an analogous formula for the case when $a(x)$ is discontinuous. For simplicity we assume that $a(x)$ has only two different values a_+ and a_-; we use the notation $D_\pm = \{x \in \mathbf{R}^d \colon a(x) = a_\pm\}$. Then the diffusion process corresponding to (0.2) is the *skew Brownian motion* $\{X(s), s \geqslant 0\}$, which may be described heuristically as

$X(\cdot) =$ Brownian motion with variance $a_\pm$, if $X(\cdot)$ is in the interior of $D_\pm$; at the boundary $\partial D_\pm$ the path $X(\cdot)$ will be reflected into $D_\pm$ with probability $a_\pm/(a_- + a_+)$ (0.3)

Henceforth we write SBM as abbreviation for skew Brownian motion. The term "skew" refers to the fact that the reflection probabilities $a_\pm/(a_- + a_+)$ are different from 1/2. In dimension $d = 1$, SBM was introduced by Itô and McKean (ref. 9, p. 221).

In the following we assume $a(x)$ to be periodic (an extension of the result to nonperiodic media is discussed in Section 7 below) and we will show that $\hat{a}$ can be represented in terms of SBM roughly speaking as follows:

$$\begin{aligned} \hat{a} = {} & (\text{arithmetical mean of } a) \\ & -(\text{dissipation of SBM, caused by the skew reflections at } \partial D_\pm) \end{aligned} \tag{0.4}$$

The first term on the right side of (0.4) corresponds to the average heat conduction in $\mathbf{R}^d \backslash \partial D_\pm$. The second term, which is related to the transmission of heat through the surface $\partial D_\pm$, will be described by the autocorrelation of the boundary process of SBM. Relation (0.4) is a *fluctuation-dissipation* relation for SBM (Green–Kubo formula).

In order to prove (0.4) we use the framework of *Dirichlet forms.*[6] Following the work of Bass and Hsu,[2] where the usual reflecting Brownian motion is considered, we derive a stochastic differential equation for SBM analogous to the Skorohod equation. From this we get (0.4) by using time reversibility of SBM as in ref. 5. After submission of this note the author obtained a preprint[13] which is also based on Dirichlet form techniques. In ref. 13 invariance principles for diffusion processes with quite general random and discontinuous coefficients are derived.

The organization of this paper is also follows. In the next section the precise definition of SBM is given an the main Theorem is stated. After sketching the plan of its proof in Section 2 and collecting some auxiliary results about SBM in Section 3, we give the detailed proof of the Theorem in Sections 4–6. In Section 7 we conclude with some comments on the Theorem and on related questions.

1. FORMULATION OF THE THEOREM

1.1. Geometry of the Medium

Let C be a compact subset of $\mathbf{R}^d$, $d \geqslant 1$, contained in the open unit cube $(0,1)^d$; the boundary ∂C is assumed to be Lipschitz continuous (cf. ref. 3, p. 491). We denote by $\nu = (\nu_1,..., \nu_d)$ the unit normal vector at the boundary ∂C, pointing to the outward of C, and by σ the surface measure on ∂C. Brackets $\langle \cdot \rangle$ denote averaging with respect to the Lebesgue measure on $[0,1]^d$. We periodize C and use the notation

$$D_- = \bigcup_{n \in \mathbf{Z}^d} (n + C)$$

$$D_+ = \mathbf{R}^d \backslash D_-$$

In particular, $\partial D_\pm$ is the surface which separates D_- form D_+.

1.2. Definition of the Effective Conductivity

Given two numbers $a_+ \geqslant 0$ and $a_- \geqslant 0$, the microscopic conductivity of the body is defined by

$$a = a_- \cdot \mathbf{1}_{D_-} + a_+ \cdot \mathbf{1}_{D_+} \tag{1.1}$$

and its arithmetical mean is given by

$$\langle a \rangle = |C| \cdot a_- + (1 - |C|) \cdot a_+ \tag{1.2}$$

where $|C|$ is the volume of C.

The effective conductivity $\hat{a} = (\hat{a}_{ij})_{1 \leqslant i,j \leqslant d}$ is defined as follows.

Case I. $a_+ > 0$, $a_- > 0$. Then

$$\begin{aligned} \hat{a}_{ij} &= \int_{[0,1]^d} dx \left[-a(x) \cdot \frac{\partial u^i}{\partial x_j} \right] \\ &= a_- \cdot \int_C dx \left(-\frac{\partial u^i}{\partial x_j} \right) + a_+ \cdot \int_{[0,1]^d \setminus C} dx \left(-\frac{\partial u^i}{\partial x_j} \right), \qquad 1 \leqslant i, j \leqslant d \end{aligned} \tag{1.3}$$

where u^i: $\mathbf{R}^d \to \mathbf{R}$ is continuous and has the following properties:

$$\Delta u^i(x) = 0, \qquad x \notin \partial D_\pm \tag{1.4}$$

$$a_- \cdot \frac{\partial u^i}{\partial v_-}(x) = a_+ \cdot \frac{\partial u^i}{\partial v_+}(x), \qquad x \in \partial D_\pm \tag{1.5}$$

$$-[u^i(x + e_k) - u^i(x)] = \delta_{ik}, \qquad 1 \leqslant k \leqslant d \tag{1.6}$$

Here we have used the notation

$$\frac{\partial g}{\partial v_+}(x) = \lim_{\varepsilon \downarrow 0} \frac{g(x + \varepsilon v) - g(x)}{\varepsilon}$$

$$\frac{\partial g}{\partial v_-}(x) = \lim_{\varepsilon \downarrow 0} \frac{g(x) - g(x - \varepsilon v)}{\varepsilon} \qquad (g\colon \mathbf{R}^d \to \mathbf{R},\ x \in \partial D_\pm)$$

Equations (1.4) and (1.5) are understood in the weak sense (cf. ref. 3, p. 503).

Case II. $a_+ > 0$, $a_- = 0$. In this case we additionally assume that $d > 1$ and that D_+ is connected. We define

$$\hat{a}_{ij} = a_+ \cdot \int_{[0,1]^d \setminus C} dx \left(-\frac{\partial u^i}{\partial x_j} \right) \tag{1.7}$$

where u^i: $\bar{D}_+ \to \mathbf{R}$ has the following properties:

$$\Delta u^i(x) = 0, \qquad x \notin \partial D_+ \tag{1.8}$$

$$\frac{\partial u^i}{\partial v_+}(x) = 0, \qquad x \in \partial D_+ \tag{1.9}$$

$$-(u^i(x + e_k) - u^i(x)) = \delta_{ik}, \qquad 1 \leqslant k \leqslant d \tag{1.10}$$

1.3. Definition of SBM

Case I. $a_+>0$, $a_->0$. In this case the definition of SBM is based on the Hilbert space

$$H^1(\mathbf{R}^d)=\{f\in L^2(\mathbf{R}^d)\colon |\nabla f|\in L^2(\mathbf{R}^d)\} \tag{1.11}$$

equipped with the scalar product

$$(f,g)_{H^1(\mathbf{R}^d)}=(f,g)+\mathscr{E}(f,g) \tag{1.12}$$

where

$$(f,g)=\int_{\mathbf{R}^d} dx\, f(x)\, g(x) \qquad \text{is the usual scalar product on } L^2(\mathbf{R}^d)$$

and the Dirichlet form $\mathscr{E}$ is given by

$$\mathscr{E}(f,g)=\tfrac{1}{2}\int_{R^d} dx\, a(x)\, \nabla f(x)\cdot\nabla g(x) \qquad \text{for} \quad f,g\in H^1(\mathbf{R}^d) \tag{1.13}$$

The SBM $\{X(s), s\geqslant 0\}$ is then defined as the Markov process which corresponds to the Dirichlet space $(H^1(\mathbf{R}^d),\mathscr{E})$, i.e., if $\{P_s, s\geqslant 0\}$ denotes the transition semigroup of SBM, then

$$\{f\in L^2(\mathbf{R}^d)\colon \lim_{t\to 0}\frac{1}{t}(f-P_tf,f)<\infty\}=H^1(\mathbf{R}^d) \tag{1.14}$$

and

$$\lim_{t\to 0}\frac{1}{t}(f-P_tf,g)=\mathscr{E}(f,g) \qquad \text{for} \quad f,g\in H^1(\mathbf{R}^d) \tag{1.15}$$

The transition density of SBM is denoted by $p_s(x,y)$ ($s>0$; $x,y\in\mathbf{R}^d$); it is defined pointwise for all $x,y\in\mathbf{R}^d$ (cf. Lemma 2 in Section 3). The SBM on the torus, i.e., the projection of $\{X(s),\ s\geqslant 0\}$ on $\mathbf{R}^d/\mathbf{Z}^d$, is denoted by $\{\bar{X}(s),\ s\geqslant 0\}$ and its transition density by $\bar{p}_s(x,y)$ ($s>0$; $x,y\in\mathbf{R}^d/\mathbf{Z}^d$).

Case II. $a_+>0, a_-=0, d>1$. In this case SBM is reflected Brownian motion in the domain D_+. Its definition is based on the Hilbert space

$$H^1(D_+)=\{f\in L^2(D_+)\colon |\nabla f|\in L^2(D_+)\} \tag{1.16}$$

equipped with the scalar product

$$(f,g)_{H^1(\mathbf{R}^d)}=(f,g)+\mathscr{E}(f,g) \tag{1.17}$$

where

$$(f, g) = \int_{D_+} dx\, f(x)\, g(x)$$

is the usual scalar product on $L^2(\mathbf{R}^d)$ and the Dirichlet form $\mathscr{E}$ is given by

$$\mathscr{E}(f, g) = \tfrac{1}{2} a_+ \cdot \int_{D_+} dx\, \nabla f(x) \cdot \nabla g(x) \qquad \text{for } \; f, g \in H^1(D_+) \tag{1.18}$$

The reflected Brownian motion in D_+ is then defined as the Markov process which corresponds to the Dirichlet space $(H^1(D_+), \mathscr{E})$. Its transition density and the transition density of the reflected Brownian motion on the torus are denoted by $p_s(x, y)$ and $\bar{p}_s(x, y)$, respectively. These densities exist pointwise (see ref. 3, Section 4).

Now we are ready to state the main result.

Theorem. In cases I and II the effective conductivity has the representation

$$\hat{a}_{ij} = \langle a \rangle \cdot \delta_{ij} - \tfrac{1}{2}(a_+ - a_-)^2 \cdot \int_0^\infty ds \iint \sigma(dy)\, \sigma(dz)\, \bar{p}_s(y, z)\, v_i(y)\, v_j(z), \qquad 1 \leqslant i, j \leqslant d \tag{1.19}$$

Remarks. (i) For any initial distribution of $X(0)$ on the cube $[0,1]^d$ one can show that $\varepsilon \cdot X(t/\varepsilon^2)$ converges weakly as $\varepsilon \downarrow 0$ to Brownian motion on $\mathbf{R}^d$ with covariance $\hat{a}$ in case I and with covariance $(1 - |C|)^{-1} \cdot \hat{a}$ in case II. Given the proof of the above Theorem, it is not difficult to obtain such an invariance principle. In the present case of periodic conductivities even the old-fashioned method of characteristic functions applies (e.g., along the lines of ref. 4). See ref. 13 for much more general invariance principles.

(ii) The Theorem was announced in ref. 10 (with a slightly different normalization of $\hat{a}$ in case II), and in case II a proof was sketched in this previous note. In the following we concentrate on case I, i.e., we assume $a_+ > 0$ and $a_- > 0$ henceforth.

2. PLAN OF THE PROOF

Let $a_+ > 0$ and $a_- > 0$ be given. According to the context, σ will denote the surface measure on ∂C or on $\partial D_\pm$, respectively. If the initial distribution of $X(0)$ is Lebesgue measure on $[0,1]^d$, expectation is denoted by E; if $X(0) = x$, expectation is denoted by E_x $(x \in \mathbb{R}^d)$.

The theorem is based on the following three propositions, whose proofs can be found in Sections 4–6.

Proposition 1. There exists a standard Brownian motion $\{B(s), s \geqslant 0\}$ on $\mathbf{R}^d$ such that

$$X(t) = X(0) + \int_0^t B(ds)\,\sqrt{a}\,(X(s)) + \tfrac{1}{2}(a_+ - a_-) \cdot \int_0^t L(ds)\,\nu(X(s)), \qquad t \geqslant 0 \tag{2.1}$$

where $L(t)$ is the boundary local time of SBM at $\partial D_\pm$.

The precise definition of the local time L is given in Lemma 2 in the next section. Equation (2.1) is the analog to the *Skorohod equation* for reflecting Brownian motion. For the proof of Proposition 1 we follow ref. 2 and use *Fukushima's decomposition* of additive functionals of regular Dirichlet processes.

Proposition 2. The asymptotic covariance of SBM equals the effective conductivity, i.e.,

$$\lim_{t \to \infty} \frac{1}{t} E[X_i(t) \cdot X_j(t)] = \hat{a}_{ij}, \qquad 1 \leqslant i, j \leqslant d \tag{2.2}$$

In order to show (2.2) we look for a periodic function F: $\mathbf{R}^d \to \mathbf{R}^d$ such that $X(s) + F(X(s))$, $s \geqslant 0$, is a *martingale*.[12] The quadratic variation of this martingale can be computed in the framework of Dirichlet spaces, which immediately will yield (2.2).

Proposition 3. The asymptotic covariance of SBM is given by

$$\lim_{t \to \infty} \frac{1}{t} E[X_i(t) \cdot X_j(t)]$$
$$= \langle a \rangle \cdot \delta_{ij} - \frac{1}{2}(a_+ - a_-)^2$$
$$\cdot \int_0^\infty ds \iint \sigma(dy)\,\sigma(dz)\,\bar{p}_s(y, z)\,\nu_i(y)\,\nu_j(z) \qquad 1 \leqslant i, j \leqslant d \tag{2.3}$$

The proof of (2.3) relies on Proposition 1 and on *the time reversibility* of SBM. The importance of symmetry properties of the underlying diffusion process was already observed in ref. 5: from these properties one obtains the vanishing of the mixed terms

$$\lim_{t \to \infty} \frac{1}{t} E\left[X_i(t) \cdot \int_0^t L(ds)\,\nu_j(X(s))\right] = 0, \qquad 1 \leqslant i, j \leqslant d$$

The Theorem follows from Propositions 2 and 3. In summary one obtains the representation (1.19) by computing the asymptotic covariance of SBM in two opposite ways: on the one hand one calculates it by an elimination of the dissipation of SBM and by computing the "costs" of this elimination (proof of Proposition 2); on the other hand one calculates the dissipation of SBM directly by using symmetry properties of SBM (Proposition 3). Equating the results obtained in these two ways yields the Theorem.

3. SOME PROPERTIES OF SBM

Lemma 1. Let $\mathbf{C}_0(\mathbf{R}^d)$ be the space of continuous functions on $\mathbf{R}^d$ with compact support, equipped with the supremum norm, and let $H^1(\mathbf{R}^d)$ be equipped with the norm (1.12). Then $H^1(\mathbf{R}^d) \cap \mathbf{C}_0(\mathbf{R}^d)$ is dense in $H^1(\mathbf{R}^d)$ and in $\mathbf{C}_0(\mathbf{R}^d)$.

Proof. Similarly as in ref. 3, Section 4.

Lemma 2. The SBM has the following properties.

(i) The transition density is defined pointwise for all $x, y \in \mathbf{R}^d$ $(s > 0)$ and $(s, x, y) \mapsto p_s(x, y)$ is continuous on $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$.

(ii) For $s > 0$ one has

$$p_s(x, y) = p_s(y, x), \qquad x, y \in \mathbf{R}^d \tag{3.1}$$

and

$$\bar{p}_s(x, y) > 0, \qquad x, y \in \mathbf{R}^d/\mathbf{Z}^d \tag{3.2}$$

(iii) There exists a unique continuous additive functional L of SBM such that for all $\alpha > 0$ and for all $x \in \mathbf{R}^d$

$$E_x \int_0^\infty L(ds)\, e^{-\alpha s} = \int_0^\infty ds\, e^{-\alpha s} \int \sigma(dy)\, p_s(x, y) \tag{3.3}$$

Proof. Similarly as in ref. 3, Section 4.

Lemma 3. There exist constants $C < \infty$ and $\gamma > 0$ such that

$$\sup_{x, y \in \mathbf{R}^d/\mathbf{Z}^d} |\bar{p}_s(x, y) - 1| \leqslant C \cdot e^{-\gamma \cdot s}, \qquad s \geqslant 1 \tag{3.4}$$

Proof. Lemma 2(i) and (ii) imply

$$\inf_{x, y \in \mathbf{R}^d/\mathbf{Z}^d} \bar{p}_1(x, y) > 0$$

From this one obtains (3.4) by standard arguments (see, e.g., ref. 4, proof of Theorem 2.15).

4. PROOF OF PROPOSITION 1

We use the notation

$$Q_k = \{x = (x_1,..., x_d) \in \mathbf{R}^d\colon -k < x_i < k \text{ for } 1 \leqslant i \leqslant d\}, \qquad k = 1, 2,... \tag{4.1}$$

and

$$T_k = \begin{cases} 0, & k = 0 \\ \inf\{t > 0\colon X(t) \in \partial Q_k\}, & k \geqslant 1 \end{cases} \tag{4.2}$$

Lemma 4. For $k \in \mathbf{N}$ there exist standard Brownian motions B^k on $\mathbf{R}^d$ and continuous additive functionals L^k of SBM such that

$$X(t \wedge T_k) - X(0) = \int_0^{t \wedge T_k} B^k(ds) \sqrt{a}\,(X(s)) + \tfrac{1}{2}(a_+ - a_-) \cdot \int_0^{t \wedge T_k} L^k(ds)\, \nu(X(s)) \tag{4.3}$$

and

$$\int_0^{t \wedge T_k} L^k(ds)\, \mathbf{1}_{\partial D_\pm}(X(s)) = L^k(t \wedge T^k), \qquad t \geqslant 0$$

for all $k \in \mathbf{N}$ with $X(0) \in Q_k$.

Proof. In three steps:

Step 1. Application of Fukushima's decomposition. Because of Lemma 1, the Dirichlet form corresponding to SBM is regular; hence we can apply the results of ref. 6, Chapter V, and we obtain the following (cf. also ref. 2, p. 1009). To any $f \in \mathbf{C}_0^2(\mathbf{R}^d)$ there exists a continuous martingale $\{M^f(t), t \geqslant 0\}$ with $M^f(0) = 0$ and a continuous additive functional $\{N^f(t), t \geqslant 0\}$ of bounded variation so that

$$f(X(t)) - f(X(0)) = M^f(t) + N^f(t), \qquad t \geqslant 0 \tag{4.4}$$

M^f has the quadratic variation

$$\langle M^f, M^f \rangle_t = \int_0^t ds\, a(X(s))\, |\nabla f|^2\,(X(s)), \qquad t \geqslant 0 \tag{4.5}$$

The measure μ^f associated to N^f is characterized by

$$\tfrac{1}{2}a_- \cdot \int_{D_-} dx\, \nabla f(x) \cdot \nabla v(x) + \tfrac{1}{2}a_+ \cdot \int_{D_+} dx\, \nabla f(x) \cdot \nabla v(x) = \int_{\mathbf{R}^d} \mu^f(dx)\, \tilde{v}(x) \tag{4.6}$$

for all $v \in H^1(\mathbf{R}^d)$, where $\tilde{v}(x)$ is a quasicontinuous modification of v. Since the boundary of $D_\pm$ is Lipschitz continuous, we can apply *Green's formula* and we obtain

$$\mu^f(dx) = -\frac{1}{2} a(x) \cdot \Delta f(x) \cdot \lambda(dx) + \frac{1}{2}(a_+ - a_-) \cdot \frac{\partial f}{\partial v}(x) \cdot \sigma(dx) \qquad (4.7)$$

where $\partial f/\partial v$ denotes the derivative of f in the direction of the normal vector v, λ is the Lesbesgue measure on $\mathbf{R}^d$, and σ the surface measure on $\partial D_\pm$.

Step 2. Application of (4.4) to the truncated coordinate functions. For $k \in \mathbf{N}$ and $1 \leqslant i \leqslant d$ we define

$$f_i^k(x) = x_i \qquad \text{for} \quad x \in Q_k \qquad (4.8)$$

and continue this function to $\mathbf{R}^d$ in such a way that it belongs to $C_0^2(\mathbf{R}^d)$. We apply the decomposition (4.4) with f_i^k ($k \in \mathbf{N}$, $1 \leqslant i \leqslant d$) and use the abbreviation

$$\begin{matrix} M^k = (M_1^k, ..., M_d^k) & \text{with} & M_i^k = M^{f_i^k}, \\ N^k = (N_1^k, ..., N_d^k) & \text{with} & N_i^k = N^{f_i^k}, \end{matrix} \qquad k \in \mathbf{N}, \quad 1 \leqslant i \leqslant d \qquad (4.9)$$

For all $k \in \mathbf{N}$ with $X(0) \in Q_k$ we obtain

$$X(t \wedge T_k) - X(0) = M^k(t \wedge T_k) + N^k(t \wedge T_k), \qquad t \geqslant 0 \qquad (4.10)$$

By means of the multidimensional analog to (4.5) and by (4.8) we have

$$\langle M_i^k, M_j^k \rangle_{t \wedge T_k} = \delta_{ij} \cdot \int_0^{t \wedge T_k} ds\, a(X(s)), \qquad t \geqslant 0 \quad (1 \leqslant i, j \leqslant d,\ k \in \mathbf{N}) \qquad (4.11)$$

For brevity we write

$$N_i^k = \tfrac{1}{2}(a_+ - a_-) \cdot \bar{N}_i^k + \tfrac{1}{2}\bar{\bar{N}}_i^k \qquad (k \in \mathbf{N},\ 1 \leqslant i \leqslant d) \qquad (4.12)$$

where the measures associated to $\bar{N}_i^k$ and $\bar{\bar{N}}_i^k$, respectively, are given by

$$\begin{cases} \bar{N}_i^k \leftrightarrow \dfrac{\partial f_i^k}{\partial v} \cdot \sigma \\ \bar{\bar{N}}_i^k \leftrightarrow -a(x) \cdot \Delta f(x) \cdot \lambda \end{cases} \qquad (4.13)$$

Denote by

$$L^k(t) = \sum_{i=1}^d \int_0^t \bar{N}_i^k(ds) \frac{\partial f_i^k/\partial v}{\sum_{j=1}^d (\partial f_j^k/\partial v)^2}(X(s)), \qquad k \in \mathbf{N} \qquad (4.14)$$

Because of (4.13), the measure associated to L^k is σ, and we have the representations

$$\bar{N}_i^k(t) = \int_0^t L^k(ds)\, \frac{\partial f_i^k}{\partial \nu}\,(X(s))$$
$$\bar{\bar{N}}_i^k(t) = -\int_0^t ds\, a(X(s)) \cdot \Delta f_i^k(X(s)) \tag{4.15}$$

Let $X(0) \in Q_k$. Since $s \leqslant T_k$ implies

$$\frac{\partial f_i^k}{\partial \nu}\,(X(s)) = \nu_i(X(s)) \qquad \text{for} \quad X(s) \in \partial D_\pm, \qquad \text{and} \qquad \Delta f_i^k(X(s)) = 0$$

we obtain from (4.10), (4.12), and (4.15)

$$X(t \wedge T_k) - X(0) = M^k(t \wedge T_k) + \tfrac{1}{2}(a_+ - a_-) \cdot \int_0^{t \wedge T_k} L^k(ds)\, \nu(X(s)), \qquad t \geqslant 0 \tag{4.16}$$

Step 3. Time change. From (4.11) we want to obtain a representation of M^k as an integral with respect to Brownian motion. For this purpose we apply the following result.

Lemma 5. Let $\mathscr{F}_t$, $t \geqslant 0$, be the filtration generated by the SBM and let

$$S_t = \inf\left\{\tau \colon \int_0^\tau ds\, a(X(s)) > t\right\}, \qquad t \geqslant 0 \tag{4.17}$$

Let furthermore R be a stopping time with respect to the filtration $\mathscr{F}_{S_t}$, $t \geqslant 0$, and $\{M(s),\ s \geqslant 0\}$ be a continuous $\mathscr{F}_t$-martingale on $\mathbf{R}^d$ with $M(0) = 0$ such that

$$\langle M_i, M_j \rangle_{S_{t \wedge R}} = \delta_{ij} \cdot (t \wedge R), \qquad t \geqslant 0 \tag{4.18}$$

Denote by $\bar{B}$ a standard Brownian motion on $\mathbf{R}^d$ which is independent of M. Then the following holds:

$$B(t) \equiv M(S_{t \wedge R}) + \bar{B}(t) - \bar{B}(t \wedge R), \qquad t \geqslant 0 \tag{4.19}$$

is a standard Brownian motion, and M can be represented in the form

$$M(t \wedge S_R) = \int_0^{t \wedge S_R} B(ds)\, \sqrt{a}\,(X(s)), \qquad t \geqslant 0 \tag{4.20}$$

Proof. For $R \equiv \infty$ Lemma 5 essentially follows from Theorem 1.6 in ref. 15, p. 170. Combining this Theorem with the result of Exercise (3.28) in ref 15, p. 149–150, we obtain that

$$M(S_{t \wedge R}) + \bar{B}(t) - \bar{B}(t \wedge R), \qquad t \geqslant 0$$

is a standard Brownian motion on $\mathbf{R}^d$. In order to see that this implies the representation (4.20), we choose to given $t > 0$ the time t' so that $t = S_{t'}$. Then we get from (4.19)

$$\begin{aligned} M(t \wedge S_R) &= M(S_{t'} \wedge S_R) = M(S_{t' \wedge R}) = B(t' \wedge R) \\ &= B\left(t' \wedge \int_0^{S_R} ds\, a(X(s))\right) \\ &= B\left(\int_0^{t \wedge S_R} ds\, a(X(s))\right) \end{aligned}$$

and the last expression can indeed be represented as an integral with respect to Brownian motion as in (4.20). This proves Lemma 5.

In order to apply Lemma 5 to (4.11) we define

$$\bar{T}_k = \int_0^{T_k} ds\, a(X(s)), \qquad k \in \mathbb{N} \tag{4.21}$$

For $k \in \mathbf{N}$ we then have

$$S_{\bar{T}_k} = T_k \qquad \text{and} \qquad S_{t \wedge \bar{T}_k} = S_t \wedge S_{\bar{T}_k} = S_t \wedge T_k$$

and from (4.11) we obtain

$$\begin{aligned} \langle M_i^k, M_j^k \rangle\, S_{t \wedge \bar{T}_k} &= \langle M_i^k, M_j^k \rangle_{S_t \wedge T_k} \\ &= \delta_{ij} \cdot \int_0^{S_t \wedge T_k} ds\, a(X(s)) \\ &= \delta_{ij} \cdot \left(\int_0^{S_t} ds\, a(X(s))\right) \wedge \left(\int_0^{T_k} ds\, a(X(s))\right) \\ &= \delta_{ij} \cdot (t \wedge \bar{T}_k), \qquad t \geqslant 0 \end{aligned} \tag{4.22}$$

Therefore Lemma 5 can be applied with the $\mathscr{F}_{S_t}$-stopping time $\bar{T}_k$ ($k \in \mathbf{N}$). Hence for any $k \in \mathbf{N}$ there exists a standard Brownian motion B^k on $\mathbf{R}^d$ such that

$$M^k(t \wedge T_k) = \int_0^{t \wedge T_k} B^k(ds)\, \sqrt{a}\,(X(s)), \qquad t \geqslant 0 \tag{4.23}$$

(4.16) and (4.23) imply (4.3), and the proof of Lemma 4 is finished.

In order to deduce Proposition 1 from (4.3) we introduce the notation

$$B(t) = \sum_{k \geq 1} (B^k(t \wedge T_k) - B^k(t \wedge T_{k-1})), \qquad t \geq 0$$

and

$$L(t) = \sum_{k \geq 1} (L^k(t \wedge T_k) - L^k(t \wedge T_{k-1})), \qquad t \geq 0$$

Then B is a standard Brownian motion and L is the continuous additive functional which is associated to σ. This follows because the Fukushima decomposition (4.4) is unique and therefore

$$B^j(t \wedge T_j) = B^k(t \wedge T_j) \qquad \text{and} \qquad L^j(t \wedge T_j) = L^k(t \wedge T_j), \qquad t \geq 0$$

for $j \leq k \in \mathbf{N}$. Since

$$B^k(t \wedge T_k) = B(t \wedge T_k) \qquad \text{and} \qquad L^k(t \wedge T_k) = L(t \wedge T_k), \qquad t \geq 0, \quad k \in \mathbf{N}$$

(4.3) becomes

$$X(t \wedge T_k) - X(0) = \int_0^{t \wedge T_k} B(ds)\, \sqrt{a}\, (X(s)) + \tfrac{1}{2}(a_+ - a_-) \cdot \int_0^{t \wedge T_k} L(ds)\, \nu(X(s)), \qquad t \geq 0$$

From this we get Proposition 1 in the limit $k \to \infty$.

5. PROOF OF PROPOSITION 2

Step 1. Tranformation of $X(t)$. Let F^i: $\mathbf{R}^d \to \mathbf{R}$ $(1 \leq i \leq d)$ satisfy the following three conditions:

$$F^i \text{ is continuous and periodic} \tag{5.1}$$

$$\Delta F^i(x) = 0, \qquad x \notin \partial D_{\pm} \tag{5.2}$$

$$a_- \cdot \left(\nu_i + \frac{\partial F^i}{\partial \nu_-}\right)(x) = a_+ \cdot \left(\nu_i + \frac{\partial F^i}{\partial \nu_+}\right)(x), \qquad x \in \partial D_{\pm} \tag{5.3}$$

We define

$$f_i(x) = x_i + F^i(x), \qquad 1 \leq i \leq d \tag{5.4}$$

and write

$$F=(F^1,\dots,F^d), \qquad f=(f_1,\dots,f_d) \tag{5.5}$$

If f belonged to $\mathbf{C}_0^2$, we could apply Itô's lemma to $f(X(t))$ and we would obtain that

$$f(X(t))=X(t)+F(X(t)), \qquad t\geqslant 0$$

is a martingale, since F was just so chosen. However, Itô's lemma cannot be used. In the next step we therefore replace f by appropriately defined functions $f^k=(f_1^k,\dots,f_d^k)$ and apply Fukushima's decomposition to these f^k $(k\in\mathbf{N})$.

Step 2. Application of Fukushima's decomposition. Let $k\in\mathbf{N}$ be given and define $f^k=(f_1^k,\dots,f_d^k)$ as follows. Using the notation $Q_k=(-k,+k)^d$ as before, we choose $\varepsilon>0$ so that

$$([-k-\varepsilon,k+\varepsilon]^d\backslash Q_k)\cap\partial D_\pm=\varnothing$$

and a $\mathbf{C}^\infty$-function φ^k: $\mathbf{R}^d\to\mathbf{R}$ so that

$$\varphi^k(x)=\begin{cases}1, & x\in Q_k\\ 0, & x\in\mathbf{R}^d\backslash[-k-\varepsilon,k+\varepsilon]^d\end{cases}$$

and let

$$f_i^k(x)=(x_i+F^i(x))\cdot\varphi^k(x), \qquad x\in\mathbf{R}^d \tag{5.6}$$

For $v\in H^1(\mathbf{R}^d)$ we then obtain

$$\mathscr{E}(f_i^k,v)=\langle\mu_i^k,\tilde{v}\rangle, \qquad 1\leqslant i\leqslant d, \quad k\in\mathbf{N} \tag{5.7}$$

where $\tilde{v}$ is a quasicontinuous modification of v and the measure μ_i^k is given by

$$\mu_i^k(dx)=-\frac{1}{2}a(x)\cdot\Delta f_i^k(x)\cdot\lambda(dx)$$
$$+\frac{1}{2}\left(a_+\cdot\frac{\partial f_i^k}{\partial v_+}-a_-\frac{\partial f_i^k}{\partial v_-}\right)(x)\cdot\sigma(dx) \tag{5.8}$$

This follows since *Green's formula* can be applied because $\partial f_i^k/\partial v_\pm$ exists on $\partial D_\pm$ and Δf_i^k exists on $\mathbf{R}^d\backslash\partial D_\pm$.

Because of (5.7) the assumption of Theorem 5.3.2. in ref. 6 is satisfied for the functions f_i^k $(1 \leqslant i \leqslant d,\ k \in \mathbf{N})$. For $k \in \mathbf{N}$, $1 \leqslant i \leqslant d$, we therefore obtain the decomposition

$$f_i^k(X(t)) - f_i^k(X(0)) = M_i^k(t) + N_i^k(t), \qquad t \geqslant 0 \tag{5.9}$$

where $M^k(t) = (M_1^k(t), \ldots, M_d^k(t))$, $t \geqslant 0$, is a continuous martingale with quadratic variation

$$\langle M_i^k, M_j^k \rangle_t = \int_0^t ds\, a(X(s))\, \nabla f_i^k(X(s)) \cdot \nabla f_j^k(X(s)), \qquad t \geqslant 0, \quad 1 \leqslant i, j \leqslant d \tag{5.10}$$

and where N_i^k is the continuous additive functional with associated measure μ_i^k. Since $\operatorname{supp} \mu_i^k \cap Q_k = \varnothing$ by definition of F, we obtain from (5.9)

$$f_i^k(X(t \wedge T_k)) - f_i^k(X(0)) = M_i^k(t \wedge T_k), \qquad t \geqslant 0 \tag{5.11}$$

if $X(0) \in Q_k$ and $T_k = \inf\{s > 0: X(s) \in \partial Q_k\}$.

Step 3. Calculation of the asymptotic covariance of SBM. By Proposition 1 it is easy to verify that there exists a constant such that

$$E\, |X(t)|^2 \leqslant \text{const} \cdot t \qquad (t \geqslant 1)$$

Because of the boundedness of F we therefore obtain

$$\begin{aligned} &\lim_{t \to \infty} \frac{1}{t} E[X_i(t) \cdot X_j(t)] \\ &\qquad = \lim_{t \to \infty} \frac{1}{t} E[\{f_i(X(t)) - f_i(X(0))\} \cdot \{f_j(X(t)) - f_j(X(0))\}] \end{aligned} \tag{5.12}$$

Furthermore, by means of Proposition 1 it is not difficult to see that $E[\sup_{0 \leqslant s \leqslant t} |X(s)|^2] < \infty$ $(t > 0)$. Hence by (5.6), (5.10), and (5.11) we get for any $t > 0$

$$\begin{aligned} &E[\{f_i(X(t)) - f_i(X(0))\} \cdot \{f_j(X(t)) - f_j(X(0))\}] \\ &\quad = \lim_{k \to \infty} E[\{f_i(X(t \wedge T_k)) - f_i(X(0))\} \cdot \{f_j(X(t \wedge T_k)) - f_j(X(0))\}] \\ &\quad = \lim_{k \to \infty} E[\{f_i^k(X(t \wedge T_k)) - f_i^k(X(0))\} \cdot \{f_j^k(X(t \wedge T_k)) - f_j^k(X(0))\}] \\ &\quad = \lim_{k \to \infty} E[M_i^k(t \wedge T_k) \cdot M_j^k(t \wedge T_k)] \\ &\quad = \lim_{k \to \infty} E\langle M_i^k, M_j^k \rangle_{t \wedge T_k} \end{aligned}$$

$$= E\int_0^t ds\, a(X(s)) \sum_{l=1}^d \left(\delta_{il}+\frac{\partial F^i}{\partial x_l}\right)\left(\delta_{jl}+\frac{\partial F^j}{\partial x_l}\right)(X(s))$$

$$= t\cdot\int_{[0,1]^d} dx\, a(x) \sum_{l=1}^d \left(\delta_{il}+\frac{\partial F^i}{\partial x_l}\right)\left(\delta_{jl}+\frac{\partial F^j}{\partial x_l}\right)(x)$$

$$= t\cdot\int_{[0,1]^d} dx\, a(x)\left(\delta_{ij}+\frac{\partial F^i}{\partial x_j}\right)(x) \tag{5.13}$$

where the last line is obtained by partial integration.

Equations (5.12) and (5.13) prove Proposition 2, since $u^i\colon \mathbf{R}^d \to \mathbf{R}$ $(1 \leqslant i \leqslant d)$, defined by

$$u^i(x) = -[x_i + F^i(x)], \qquad x \in \mathbf{R}^d$$

satisfies (1.4)–(1.6), and

$$\int_{[0,1]^d} dx\, a(x)\left(\delta_{ij}+\frac{\partial F^i}{\partial x_j}\right)(x)$$

$$= \int_{[0,1]^d} dx\left[-a(x)\cdot\frac{\partial u^i}{\partial x_j}\right] = \hat{a}_{ij} \qquad (1 \leqslant i, j \leqslant d)$$

6. PROOF OF PROPOSITION 3

Using the abbreviation

$$A_i(t) = \int_0^t L(ds)\, v_i(X(s)), \qquad t \geqslant 0, \quad 1 \leqslant i \leqslant d \tag{6.1}$$

we obtain from Proposition 1

$$E\left[\int_0^t B_i(ds)\sqrt{a}\,(X(s))\cdot\int_0^t B_j(ds)\sqrt{a}\,(X(s))\right]$$

$$= E[\{[X_i(t)-X_i(0)] - \tfrac{1}{2}(a_+ - a_-)\cdot A_i(t)\}$$

$$\cdot\{[X_j(t)-X_j(0)] - \tfrac{1}{2}(a_+ - a_-)\cdot A_j(t)\}], \qquad 1 \leqslant i, j \leqslant d \tag{6.2}$$

In order to compute the right-hand side of (6.2) we use[5]

$$E[\{X_i(t)-X_i(0)\}\cdot A_j(t)] = 0, \qquad t \geqslant 0, \quad 1 \leqslant i, j \leqslant d \tag{6.3}$$

To see (6.3), we consider the time-reversed process $\hat{X}(s) = X(t-s)$, $0 \leqslant s \leqslant t$ $(t > 0$ fixed), the corresponding local time $\hat{L}(s)$, $0 \leqslant s \leqslant t$, and

$\hat{A}(t)=\int_0^t \hat{L}(ds)\, v(\hat{X}(s))$. By Lemma 2(ii) the transition density of SBM is symmetric; hence

$$\begin{aligned}E[\{X_i(t)-X_i(0)\}\cdot A_j(t)] &= E[\{\hat{X}_i(t)-\hat{X}_i(0)\}\cdot \hat{A}_j(t)]\\ &= E[\{X_i(0)-X_i(t)\}\cdot \hat{A}_j(t)]\\ &= E[\{X_i(0)-X_i(t)\}\cdot A_j(t)]\end{aligned}$$

since $A(t)$ is invariant by time reversal. This shows (6.3). From (6.2) and (6.3) we obtain

$$\begin{aligned}&E[\{X_i(t)-X_i(0)\}\cdot\{X_j(t)-X_j(0)\}]\\ &\quad = E\left[\int_0^t B_i(ds)\sqrt{a}\,(X(s))\cdot\int_0^t B_j(ds)\sqrt{a}\,(X(s))\right]\\ &\qquad -\tfrac{1}{4}(a_+-a_-)^2\cdot E[A_i(t)\cdot A_j(t)] \end{aligned}\tag{6.4}$$

and therefore

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}E[X_i(t)\cdot X_j(t)]\\ &\quad =\delta_{ij}\cdot\int_{[0,1]^d} dx\, a(x)-\frac{1}{4}(a_+-a_-)^2\cdot\lim_{t\to\infty}\frac{1}{t}E[A_i(t)\cdot A_j(t)]\\ &\quad =\langle a\rangle\cdot\delta_{ij}-\frac{1}{4}(a_+-a_-)^2\cdot\lim_{t\to\infty}\frac{1}{t}E[A_i(t)\cdot A_j(t)]\end{aligned}\tag{6.5}$$

The transition density of SBM is symmetric and has Lebesgue measure as invariant measure. Hence

$$\begin{aligned}&E[A_i(t)\cdot A_j(t)]\\ &\quad =\int_{[0,1]^d} dx\int_0^t ds_1\int\sigma(dy)\,\bar{p}_{s_1}(x,y)\,v_i(y)\cdot\int_{s_1}^t ds_2\int\sigma(dz)\,\bar{p}_{s_2-s_1}(y,z)\,v_j(z)\\ &\qquad +\int_{[0,1]^d} dx\int_0^t ds_1\int\sigma(dy)\,\bar{p}_{s_1}(x,y)\,v_j(y)\\ &\qquad \cdot\int_{s_1}^t ds_2\int\sigma(dz)\,\bar{p}_{s_2-s_1}(y,z)\,v_i(z)\\ &\quad =2\cdot\int\sigma(dy)\,v_i(y)\int_0^t ds\,(t-s)\int\sigma(dz)\,\bar{p}_s(y,z)\,v_j(z)\\ &\quad =2\cdot\int\sigma(dy)\,v_i(y)\int_0^t ds\,(t-s)\int\sigma(dz)\,[\bar{p}_s(y,z)-1]\,v_j(z)\end{aligned}\tag{6.6}$$

Because of Lemma 3, Lebesgue's theorem of dominated convergence can be applied, and we obtain

$$\lim_{t \to \infty} \frac{1}{t} E[A_i(t) \cdot A_j(t)] = 2 \cdot \int_0^\infty ds \iint \sigma(dy)\, \sigma(dz)\, \bar{p}_s(y, z)\, v_i(y)\, v_j(z) \quad (6.7)$$

Equations (6.5) and (6.7) prove Proposition 3.

7. REMARKS AND OPEN PROBLEMS

7.1. Extension of the Theorem to General Ergodic Media

We believe that the Theorem can be extended from periodic to general ergodic media. More precisely, let D_- be a random closed subset of $\mathbf{R}^d$ which is spatially stationary and ergodic. Assume further that the surface which separates D_- and $D_+ = \mathbf{R}^d \backslash D_-$ is smooth and without self-intersections. Let the microscopic conductivity be given by $a = a_- \cdot \mathbf{1}_{D_-} + a_+ \cdot \mathbf{1}_{D_+}$ with $a_\pm > 0$. The SBM, moving in $D_\pm$ with variance $a_\pm$, is defined similarly as in the periodic case. We denote its (random) transition density by $p_s(x, y)$ ($s > 0$; $x, y \in \mathbf{R}^d$). Then the analog to the representation formula (1.19) should be

$$\hat{a}_{ij} = \langle a \rangle \cdot \delta_{ij} - \tfrac{1}{2} \gamma \cdot (a_+ - a_-)^2 \cdot \int_0^\infty ds \left\langle \int \sigma(dy)\, p_s(0, y)\, v_i(0) \cdot v_j(y) \right\rangle_0 \quad (7.1)$$

where $\langle \cdot \rangle$ denotes averaging with respect to the medium, $\langle \cdot \rangle_0 = \langle \cdot \,|\, 0 \in \partial D_\pm \rangle$ is the Palm measure of the measure $\langle \cdot \rangle$, and

$$\gamma = \lim_{K \uparrow \mathbf{R}^d} \frac{|\partial D_\pm \cap K|}{|K|}$$

is the specific surface of the boundary $\partial D_\pm$.

7.2. Representation of $\hat{a}$ in Terms of the Boundary Process

We consider the SBM as in Section 1 and denote by

$$\tau(t) = \sup\{s \geqslant 0 \colon L(s) \leqslant t\}, \qquad t \geqslant 0$$

the right continuous inverse of the boundary local time L. The process $\{\xi(s),\ s \geqslant 0\}$, defined by

$$\xi(s) = \bar{X}(\tau(s)), \qquad s \geqslant 0 \quad (7.2)$$

is the *boundary process* of SBM on the torus. It is plausible that the representation formula (1.19) can be expressed in terms of the boundary process as follows:

$$\hat{a}_{ij} = \langle a \rangle \cdot \delta_{ij} - \tfrac{1}{2}(a_+ - a_-)^2 \cdot \int_0^\infty ds\, E_\sigma[v_i(\xi(0)) \cdot v_j(\xi(s))] \tag{7.3}$$

To see this, we use in (1.19) the transformation $s = \tau(s')$, $L(s) = L(\tau(s')) = s'$, and get

$$\begin{aligned}
&\int_0^\infty ds \iint \sigma(dy)\, \sigma(dz)\, \bar{p}_s(y, z)\, v_i(y)\, v_j(z) \\
&\quad = \lim_{t \to \infty} \int \sigma(dy)\, v_i(y) \int_0^t ds \int \sigma(dz)\, \bar{p}_s(y, z)\, v_j(z) \\
&\quad = \lim_{t \to \infty} \int \sigma(dy)\, v_i(y)\, E_y \int_0^t L(ds)\, v_j(\bar{X}(s)) \\
&\quad = \lim_{t \to \infty} \int \sigma(dy)\, v_i(y)\, E_y \int_0^\infty ds\, v_j(\bar{X}(\tau(s)))\, \mathbf{1}_{\{\tau(s) \leqslant t\}} \\
&\quad = \lim_{t \to \infty} \int_0^\infty ds\, E_\sigma[v_i(\bar{X}(0)) \cdot v_j(\bar{X}(\tau(s)))\, \mathbf{1}_{\{\tau(s) \leqslant t\}}]
\end{aligned}$$

If in the last line the interchange of the limit with integration could be justified, we would obtain (7.3). We have not yet shown the good mixing properties of the boundary process, which seem to be required for a rigorous proof of (7.3).

7.3. Definition of SBM by Means of Excursions

The intuitive description of SBM given in the introduction [cf. (0.3)] can be made precise by means of *excursions*. To show this, we adapt the corresponding construction of reflecting Brownian motion in ref. 8 and sketch the main steps of a pathwise construction of SBM. We assume that the given set C has a $\mathbf{C}^3$-boundary. We write $C_- = C$, $C_+ = (\mathbf{R}^d/\mathbf{Z}^d)\backslash C_-$ and construct the SBM $\{\bar{X}(s),\ s \geqslant 0\}$ on the torus $(\mathbf{R}^d/\mathbf{Z}^d)$ in three steps as follows:

Step 1. Construction of the boundary process $\xi(s)$, $s \geqslant 0$. The boundary process $\{\xi(s),\ s \geqslant 0\}$ is a pure jump process on $\partial C_\pm$. In order to define its generator A, we consider a continuous function $f\colon \partial C_{\pm} \to \mathbf{R}$ and the corresponding Dirichlet problem on the torus

$$\begin{cases} \Delta u(x) = 0, & x \in (\mathbf{R}^d/\mathbf{Z}^d)\backslash \partial C_{\pm} \\ u(x) = f(x), & x \in \partial C_{\pm} \end{cases} \tag{7.4}$$

and denote the solution to (7.4) by u_f.

Then

$$Af = \frac{1}{2}\left[a_{+} \cdot \frac{\partial}{\partial v_{+}} (u_f) - a_{-} \cdot \frac{\partial}{\partial v_{-}} (u_f) \right] \tag{7.5}$$

We assume ξ is right continuous and denote the set of jump times by

$$J = \{ s \geqslant 0\colon \xi(s^{-}) \neq \xi(s) \} \tag{7.6}$$

Step 2. Description of the excursions $\{e_s, s \in J\}$. According to ref. 8, there exist two point processes of excursions $\{e_s^{+}, s \geqslant 0\}$ and $\{e_s^{-}, s \geqslant 0\}$, respectively, which correspond to reflecting Brownian motion in the set C_{+} (with variance a_{+}) and in the set C_{-} (with variance a_{-}), respectively. For $s \in J$ we choose random signs $\rho(s) \in \{\pm\}$ (independently of each other, of the boundary process, and of the excursions) according to

$$P(\rho(s) = \pm) = \frac{a_{\pm}}{a_{-} + a_{+}} \tag{7.7}$$

The excursions of SBM are then defined by

$$e_s = e_s^{\rho(s)}, \qquad s \in J \tag{7.8}$$

Step 3. Definition of SBM by means of the boundary process and of the excursions. Denote by $|e_s|$ the lifetime of the excursion e_s ($s \in J$), define

$$\tau(t) = \sum_{s \in J \cap (0, t]} |e_s|, \qquad t \geqslant 0$$

and let $L(t)$, $t \geqslant 0$, be the inverse of τ. The SBM on the torus is then defined by

$$\bar{X}(t) = \begin{cases} \xi(0), & t = 0 \\ \xi(L(t)), & L(t) \notin J \\ e_{L(t)}[t - \tau(L(t)^{-})], & L(t) \in J \end{cases} \tag{7.9}$$

This process is equivalent to the process which was defined in Section 2. The equivalence can be shown as in ref. 8 via the martingale problem for

SBM (see ref. 1 for the martingale approach to diffusion processes with singular coefficients).

We conclude with a remark on the nature of the stochastic differential equation (2.1). If the separating surface ∂C is smooth, SBM can be constructed via excursions as indicated above. This suggests that in the smooth case Eq. (2.1) should hold in a strong sense, i.e., SBM should satisfy (2.1) with a standard Brownian motion $\{B(s),\ s \geqslant 0\}$ given in advance, in contrast to the weak result of Proposition 1.

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REFERENCES

1. S. V. Anulova, Diffusion processes with singular characteristics, in *Stochastic Differential Systems. Filtering and Control*, B. Grigelionis, ed. (Springer, Berlin, 1980), pp. 264–269.
2. R. F. Bass and P. Hsu, The semimartingale structure of reflecting Brownian motion, *Proc. Am. Math. Soc.* **108**:1007–1010 (1990).
3. R. F. Bass and P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains, *Ann. Prob.* **19**:486–508 (1991).
4. J. R. Baxter and G. A. Brosamler, Energy and the law of the iterated logarithm, *Math. Scand.* **38**:115–136 (1976).
5. A. De Masi, P. A. Ferrari, S. Goldstein, and D. W. Wick, An invariance principle for reversible Markov processes: Applications to random motions in random environments, *J. Stat. Phys.* **55**:787–855 (1989)
6. M. Fukushima, *Dirichlet Forms and Markov Processes* (North-Holland, Amsterdam, 1980).
7. K. Golden and G. Papanicolaou, Bounds for effective parameters of heterogeneous media by analytic continuation, *Comm. Math. Phys.* **90**:473–491 (1983).
8. P. Hsu, On excursions of reflecting Brownian motion, *Trans. Am. Math. Soc.* **296**:239–264 (1986).
9. K. Itô and H. P. McKean, Brownian motions on a half line, *Illinois J. Math.* **7**:181–231 (1963).
10. R. Lang, Une représentation probabiliste de la conductivité effective, *C. R. Acad. Sci. Paris* **317**(Ser. I):967–969 (1993).
11. K. Lichtenecker, Die Dielektrizitätskonstante natürlicher und künstlicher Mischkörper, *Phys. Z.* **27**:115–158 (1926).
12. H. Osada, Homogenization of diffusion processes with random stationary coefficients, in *Proceedings of the 4th Japan–USSR Symposium on Probability Theory* (Springer, Berlin, 1983), pp. 507–517.

13. H. Osada and T. Saitoh, An invariance principle for non-symmetric Markov processes and reflecting diffusions in random domains, *Prob. Theory Related Fields* (1995), to appear.
14. G. Papanicolaou and S. R. S. Varadhan, Boundary value problems with rapidly oscillating coefficients, in *Random Fields*, J. Fritz, J. L. Lebowitz, and D. Szász, eds. (North-Holland, Amsterdam, 1981), pp. 835–853.
15. D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, (Springer, Berlin, 1991).
16. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer, Berlin, 1991).
17. V. V. Zhikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals* (Springer, Berlin, 1994).